PMM U.S.S.R.,Vol.53,No.1,pp.81-84,1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

GREEN'S MATRIX OF THE PLANE PROBLEM OF ELASTICITY THEORY FOR AN ORTHOTROPIC STRIP*

I.M. DOLGOVA and YU.A. MEL'NIKOV

Green's matrix for a homogeneous and orthotropic half-strip clamped between two absolutely rigid half-planes is constructed. The boundary conditions on all sides correspond to frictionless contact. The equations of equilibrium in displacements formulated for the case with mass forces are solved by Fourier transform methods. The final results are represented by relatively simple formulas. The Green's matrix elements, which in physical terms represent the displacements of the half-strip points under the action of a concentrated force, are expressed in terms of elementary functions. Numerical results are given for the case with a transverse concentrated force. Hence it is shown that the algorithm for constructing Green's functions and matrices for a mixed boundary-value problem of elasticity theory with an isotropic strip /1/ can be extended to some plane problems of elasticity theory for orthotropic materials.

1. Consider the elastic equilibrium of the homogeneous strip $(-\infty < x < +\infty,\ 0 \leqslant y \leqslant b)$ with principal directions of orthotropy coinciding with the coordinate axes x and y; E_1 and E_2 are Young's moduli of the first kind, ν_1 and ν_2 are Poisson's ratios, and G is the shear modulus.

The system of equations of equilibrium in displacements is written in the form

$$\Lambda\,(\partial^2/\partial x^2,\ \partial^2/\partial y^2,\ \partial^2/\partial x\partial y,\ A_{11},\ A_{12},\ A_{22},\ A_{66})\,\mathbf{U}\,(x,\ y) = \mathbf{F}\,(x,\ y) \tag{1.1}$$

Here $\mathbf{U} = \mathbf{U}\,(x,\ y)$ is the vector of displacements of the strip points and $\mathbf{F} = \mathbf{F}\,(x,\ y)$ is the vector of mass forces. The elements of the matrix $\Lambda = (\Lambda^{ij})_{2,2}$ are written in the form (we follow the notation of /2/)

$$\Lambda^{11} \equiv A_{11}\partial^2/\partial x^2 + A_{66}\partial^2/\partial y^2,\quad \Lambda^{22} \equiv A_{66}\partial^2/\partial x^2 + A_{22}\partial^2/\partial y^2$$
$$\Lambda^{12} = \Lambda^{21} \equiv (A_{12} + A_{66})\,\partial^2/\partial x\partial y$$
$$A_{11} = \frac{E_1}{1-\nu_1\nu_2},\quad A_{12} = \frac{\nu_1 E_2}{1-\nu_1\nu_2},\quad A_{22} = \frac{E_2}{1-\nu_1\nu_2},$$
$$A_{66} = G$$

Assume that the stress-strain state being analysed is symmetrical about the y axis. Then clearly the components $u\,(x,\ y)$ and $v\,(x,\ y)$ of the vector $\mathbf{U}\,(x,\ y)$ for $x = 0$ should satisfy the relationships

$$u = 0,\quad \partial v/\partial x = 0$$

We stipulate that these components must vanish as $x \to \infty$, and on the edges $y = 0,\ y = b$ we consider the conditions of contact of the strip with absolutely rigid half-planes without friction and without lag,

$$v = 0,\quad \partial u/\partial y = 0 \tag{1.2}$$

We represent the solution of the problem and the mass force vector $\mathbf{F}\,(x,\ y)$ by the expansions

$$\mathbf{U}\,(x, y) = \sum_{n=0}^{\infty} Q_n\,(y)\,U_n\,(x),\quad \mathbf{F}\,(x, y) = \sum_{n=0}^{\infty} Q_n\,(y)\,F_n\,(x) \tag{1.3}$$
$$Q_n\,(y) \equiv \begin{Vmatrix} \cos\nu y & 0 \\ 0 & \sin\nu y \end{Vmatrix},\quad \nu = \frac{n\pi}{b}$$

which obviously enables us to satisfy conditions (1.2) and for the components $u_n\,(x)$ and $v_n\,(x)$ of the vector $\mathbf{U}_n\,(x)$ leads to a system of linear ordinary differential equations

$$L_n\,(d^2/dx^2,\ A_{11},\ A_{12},\ A_{22},\ A_{66})\,\mathbf{U}_n\,(x) = \mathbf{F}_n\,(x) \tag{1.4}$$

*Prikl.Matem.Mekhan.,53,1,102-106,1989

with the boundary conditions

$$x = 0 \quad u_n = 0, \quad v_n' = 0; \quad x \to \infty, \quad u_n = 0, \quad v_n = 0 \tag{1.5}$$

The elements L_n^{ij} of the matrix L_n are given by the expressions

$$L_n^{11} \equiv A_{11}d^2/dx^2 - A_{66}\nu^2, \quad L_n^{22} \equiv A_{66}d^2/dx^2 - A_{22}\nu^2$$
$$L_n^{12} = -L_n^{21} \equiv (A_{12} + A_{66})\,\nu d/dx$$

The fundamental system of solutions of the homogeneous system corresponding to (1.4) consists of the vectors

$$\mathbf{U}_n^*(x, \ \pm p), \quad \mathbf{U}_n^*(x, \ \pm q) \tag{1.6}$$

Here

$$\mathbf{U}_n^*(x, p) = \left\| \begin{matrix} \dfrac{A_{12} + A_{66}}{\nu(A_{66} - A_{11}p^2)} e^{\nu px} \\ (\nu p)^{-1} e^{\nu px} \end{matrix} \right\|$$
$$p, q = \left(\frac{A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66} \pm B^{1/2}}{2A_{11}A_{66}} \right)^{1/2}$$
$$B = (A_{12}^2 - A_{11}A_{22})^2 + 4(A_{12}^2 - A_{11}A_{22})A_{66}(A_{12} + A_{66})$$

Applying the procedure of Lagrange's method of variation of arbitrary constants, we obtain a general solution of system (1.4) in the form

$$\mathbf{U}_n(x) = \int_0^x S_n(x, \xi)\, \mathbf{F}_n(\xi)\, d\xi + P_n(x)\, C_n \tag{1.7}$$

The elements $S_n^{ij}(x, \xi)$ of the matrix $(S_n(x, \xi))_{2,2}$ are given by the expressions

$$S_n^{11}(x, \ \xi) = (pD_q \operatorname{sh} \nu p(x - \xi) - \ldots)/(A_{11}R)$$
$$S_n^{12}(x, \ \xi) = \mu(\operatorname{ch} \nu p(x - \xi) + \ldots)/(A_{66}(D_p - D_q))$$
$$S_n^{21}(x, \ \xi) = D_pD_q(\operatorname{ch} \nu p(x - \xi) - \ldots)/(\mu A_{11}R)$$
$$S_n^{22}(x, \ \xi) = (qD_p \operatorname{sh} \nu p(x - \xi) - \ldots)/(A_{66}pq(D_p - D_q))$$
$$\mu = A_{12} + A_{66}, \quad D_p = A_{66} - A_{11}p^2, \quad D_q = A_{66} - A_{11}q^2,$$
$$R = p^2D_q - q^2D_p$$

Here and henceforth, ellipsis stands for the quantity corresponding to the previous term with p replaced by q and q by p; $P_n(x) = (\mathbf{U}_n^*)_{2,4}$ is a matrix whose columns are the vectors (1.6).

Once the boundary conditions (1.5) have been satisfied, the column matrix of the arbitrary constants C_n is determined by the integral

$$C_n = \int_0^\infty W_n(\xi)\, \mathbf{F}_n(\xi)\, d\xi$$

which, substituted in (1.7), gives

$$\mathbf{U}_n(x) = \int_0^\infty G_n(x, \xi)\, \mathbf{F}_n(\xi)\, d\xi \tag{1.8}$$
$$G_n(x, \xi) = \begin{cases} S_n(x, \xi) + P_n(x)\, W_n(\xi), & x > \xi \\ P_n(x)\, W_n(\xi), & x < \xi \end{cases}$$

The kernel $G_n(x, \xi)$ is Green's matrix of the boundary-value problem (1.4), (1.5) and its elements are

$$G_n^{11}(x, \xi) = \begin{cases} -K_{11}\nu^{-1}(pD_q e^{-\nu px} \operatorname{sh} \nu p\xi - \ldots), & x > \xi \\ -K_{11}\nu^{-1}(pD_q e^{-\nu p\xi} \operatorname{sh} \nu px - \ldots), & x < \xi \end{cases}$$
$$G_n^{12}(x, \xi) = \begin{cases} K_{12}\nu^{-1}(e^{-\nu px} \operatorname{ch} \nu p\xi - \ldots), & x > \xi \\ -K_{12}\nu^{-1}(e^{-\nu p\xi} \operatorname{sh} \nu px - \ldots), & x < \xi \end{cases}$$
$$G_n^{21}(x, \xi) = \begin{cases} K_{21}\nu^{-1}(e^{-\nu px} \operatorname{sh} \nu p\xi - \ldots), & x > \xi \\ -K_{21}\nu^{-1}(e^{-\nu p\xi} \operatorname{ch} \nu px - \ldots), & x < \xi \end{cases}$$

$$G_n^{22}(x,\xi)=\begin{cases}K_{22}\nu^{-1}(pD_q e^{-\nu q x}\operatorname{ch}\nu q\xi-\ldots), & x>\xi\\ K_{22}\nu^{-1}(pD_q e^{-\nu q\xi}\operatorname{ch}\nu q x-\ldots), & x<\xi\end{cases}$$

$$K_{11}=\frac{1}{A_{11}R},\quad K_{12}=\frac{\mu}{A_{66}(D_p-D_q)},$$

$$K_{21}=\frac{D_pD_q}{\mu A_{11}R},\quad K_{22}=\frac{1}{A_{66}pq(D_p-D_q)}$$

Now applying the Fourier-Euler formula for $\mathbf{F}_n(\xi)$ in (1.8) and then substituting $\mathbf{U}_n(x)$ into (1.3), we obtain

$$\mathbf{U}(x,y)=\int_0^\infty\int_0^b G(x,y;\xi,\eta)\,\mathbf{F}(\xi,\eta)\,d\xi\,d\eta \tag{1.9}$$

$$G(x,y;\xi,\eta)=\frac{1}{b}\sum_{n=0}^{\infty}\varepsilon_n Q_n(y)\,G_n(x,\xi)\,Q_n(\eta) \tag{1.10}$$

$$\varepsilon_n=\begin{cases}1, & n=0\\ 2, & n>0\end{cases}$$

Since the original boundary-value problem of elasticity theory is uniquely solvable for an orthotropic half-strip and its solution has a known representation by a mass potential, we conclude that the kernel (1.10) of integral (1.9) is the required Green's matrix of our problem.

Now noting that for $t<1$ and $0\leqslant y\leqslant 2\pi$ we have

$$\sum_{n=1}^{\infty}\frac{t^n\cos ny}{n}=-\frac{1}{2}\ln(1-2t\cos y+t^2)$$

$$\sum_{n=1}^{\infty}\frac{t^n\sin ny}{n}=\operatorname{arctg}\frac{t\sin y}{1-t\cos y}$$

we perform summation in the elements of the Green's matrix and obtain

$$G_{11}(x,y;\xi,\eta)=\frac{K_{11}}{4\pi}\left[pD_q\ln\frac{E(p,x,y;-\xi,\eta)\,E(p,x,y;-\xi,-\eta)}{E(p,x,y;\xi,\eta)\,E(p,x,y;\xi,-\eta)}-\ldots\right]$$

$$G_{12}(x,y;\xi,\eta)=\frac{K_{12}}{2\pi}[\operatorname{arctg}M(p,x,y;\xi,\eta)-\ldots$$
$$+\delta\operatorname{arctg}M(p,x;y;-\xi,\eta)-\ldots+\operatorname{arctg}M(q,x,y;\xi,-\eta)-\ldots$$
$$+\delta\operatorname{arctg}M(q,x,y;-\xi,-\eta)-\ldots]$$

$$G_{21}(x,y;\xi,\eta)=\frac{K_{21}}{2\pi}[\operatorname{arctg}M(q,x,y;\xi,\eta)-\ldots$$
$$+\delta\operatorname{arctg}M(p,x,y;-\xi,\eta)-\ldots+\operatorname{arctg}M(q,x,y;\xi,-\eta)-\ldots$$
$$+\delta\operatorname{arctg}M(p,x,y;-\xi,-\eta)-\ldots]$$

$$G_{22}(x,y;\xi,\eta)=\frac{K_{22}}{4\pi}\left[pD_q\ln\frac{E(p,x,y;\xi,-\eta)\,E(p,x,y;-\xi,-\eta)}{E(p,x,y;\xi,\eta)\,E(p,x,y;-\xi,\eta)}-\ldots\right]$$

$$\delta=\begin{cases}1, & x>\xi\\ -1, & x<\xi\end{cases}$$

$$E(p,x,y;\xi,\eta)=1-2\exp\left(-\frac{\pi}{b}p|x+\xi|\right)\cos\left[\frac{\pi}{b}(y+\eta)\right]+$$
$$\exp\left(-\frac{2\pi}{b}p|x+\xi|\right)$$

$$M(p,x,y;\xi,\eta)=\exp\left(-\frac{\pi}{b}p|x+\xi|\right)\sin\left[\frac{\pi}{b}(y+\eta)\right]\times$$
$$\left(1-\exp\left(-\frac{\pi}{b}p|x+\xi|\right)\cos\left[\frac{\pi}{b}(y+\eta)\right]\right)^{-1}$$

We can verify that this construction satisfies all the defining properties of Green's matrix.

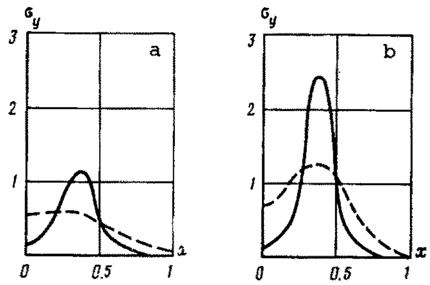
2. Let us now present some results of the application of this Green's matrix. Using the mechanical interpretation of Green's matrix, we see that the components $G_{12}(x,y;\xi,\eta)$, $G_{22}(x,y;\xi,\eta)$ are the displacements $u(x,y)$ and $v(x,y)$ of the point (x,y) of the given half-strip produced by a unit mass force applied at the point (ξ,η) and directed parallel to the y axis. Applying well-known formulas /2/, we obtain for the corresponding normal stresses

$$\sigma_x = A_{11}\partial G_{12}/\partial x + A_{12}\partial G_{22}/\partial y$$
$$\sigma_y = A_{12}\partial G_{12}/\partial x + A_{22}\partial G_{22}/\partial y$$

which in our case take the form

$$\sigma_x = \frac{A_{12}+A_{11}p^2}{2bA_{11}p\,(q^2-p^2)}\,[\Phi(p,x,y;-\xi,-\eta)+\Phi(p,x,y;\xi,-\eta)-$$
$$\Phi(p,x,y;-\xi,\eta)-\Phi(p,x,y;\xi,\eta)]+\ldots$$
$$\sigma_y = \frac{p^2(A_{12}^2-A_{11}A_{22}+A_{21}A_{66})+A_{22}A_{66}}{2bA_{11}A_{66}p\,(q^2-p^2)}\,[\Phi(p,x,y;-\xi,-\eta)+$$
$$\Phi(p,x,y;\xi,-\eta)-\Phi(p,x,y;-\xi,\eta)-\Phi(p,x,y;\xi,\eta)]+\ldots$$
$$\Phi(p,x,y;\xi,\eta)=\exp\left(-\frac{\pi}{b}p\,|x+\xi|\right)\times$$
$$\sin\left[\frac{\pi}{b}(y+\eta)\right][E(p,x,y;\xi,\eta)]^{-1}$$

Fig.1 plots the stresses σ_y on the upper edge (a) and lower edge (b) of the half-strip of width $b=1$ due to a unit force concentrated at the point (0.35, 0.65). The solid curves correspond to the stress components in the orthotropic case ($E_1 = 0.59\times10^9$ N/m^2, $E_2 = 1.18\times10^9$ N/m^2, $\nu_1 = 0.036$, $\nu_2 = 0.071$, and $G = 0.07\times10^9$ N/m^2), the broken curves correspond to the isotropic case ($E = 0.59\times10^9$ N/m^2), $\nu = 0.036$).



Note that the existence of Green's matrix derived above enables one to consider the action of any system of concentrated or distributed forces (with a known distribution) acting on an orthotropic strip. This matrix also can be used to develop algorithms for solving a number of applied problems using existing versions of the potentials method /3, 4/.

REFERENCES

1. DOLGOVA I.M. and MEL'NIKOV YU.A., Construction of Green's functions and methods for equations and systems of elliptic type, PMM, 42, 4, 1978.
2. LEKHNITSKII S.G., Elasticity Theory of Anisotropic Bodies, Nauka, Moscow, 1977.
3. MEL'NIKOV YU.A., An elastic strip with a regular series of rigidly fringed holes of complex outline, Dokl. Akad. Nauk, UkrSSR, Ser. A, 7, 1977.
4. KOSHNAREVA V.A., MEL'NIKOV YU.A. and PREOBRAZHENSKII I.N., The solution of plane contact problems of elasticity theory by Green's function method, Izv. Akad. Nauk, SSSR, MTT, 4, 1987.

Translated by Z.L.